The Universality and stability for a dilute Bose gas with a Feshbach resonance

Yu-Li Lee[1,*] and Yu-Wen Lee[2,†]

[1]*Department of Physics, National Changhua University of Education, Changhua, Taiwan, R.O.C.*
[2]*Department of Physics, Tunghai University, Taichung, Taiwan, R.O.C.*

We study the bosonic atoms with a wide Feshbach resonance at zero temperature in terms of the renormalization group. We indicate that this system will always collapse in the dilute limit. On the side with a positive scattering length, the atomic superfluid is an unstable local minimum in the dilute limit and it determines the thermodynamics of this system within its lifetime. We calculate the equilibrium properties at zero temperature in the unitary regime. They exhibit universal scaling forms in the dilute limit due to the presence of a nontrivial zero temperature, zero density fixed point. Moreover, we find that the $T = 0$ thermodynamics of this system in the unitary limit is exactly identical to the one for an ideal Fermi gas.

PACS numbers: 03.75.-b, 67.85.Jk, 67.10.Ba,

I. INTRODUCTION

The bosonic atoms with Feshbach resonances have been one of the exciting ultrocold systems studied so far. On the side with a negative scattering length, the controlled collapsing and exploding dynamics has been observed[1]. On the other side where the scattering length becomes positive, molecules can be produced and this system becomes a mixture of atoms and molecules. Despite the reduced lifetime of molecules in the unitary regime due to the enhanced three-body recombination[2], quite remarkable progress has been made and the properties of the strongly repulsive ultracold bosonic atoms have also been studied experimentally[3].

A mean-field phase diagram for the Bose gas with an s-wave Feshbach resonance has been proposed[4]. At zero temperature, there are two phases: the atomic superfluid (ASF) in which both atoms and molecules condense and the molecular superfluid (MSF) in which only the molecules condense. Hence, there is a quantum phase transition between the ASF and MSF phases[4,5]. The stability of these phases at $T = 0$ was examined within the mean-field level[6]. Recently, a Nozières-Schmitt-Rink formalism was developed to study the finite temperature properties of this system[7]. The ground state of the cold Bose gas with a large positive scattering length has also been studied by the variational wavefunction approach[8,9]. Near the Feshbach resonance, various physical quantities were found to exhibit universal scaling forms[9].

Although the idea of the Feshbach resonance has its root in the few-body physics, it was shown that, for a dilute quantum gas, it is possible to cast the low-energy physics near the Feshbach resonance and the associated universal properties into the framework of the renormalization group (RG) due to the presence of a nontrivial zero-density fixed point[10]. Bearing this picture in mind, the low-energy properties of a dilute Fermi gas with a wide Feshbach resonance in the unitary regime can be computed either by an ϵ expansion[11] ($\epsilon = 4 - d$) or a large N expansion[10,12]. Here we will extend the idea in Ref. 10 to the dilute Bose gas with a wide Feshbach resonance. In contrast to the case of fermionic atoms, the self-interactions between atoms and molecules must be included in order to guarantee the stability of the uniform phases although these operators are irrelevant in the sense of RG. This point was also emphasized in Ref. 6. The resulting stability conditions impose a lower bound on the value of the chemical potential for atoms. Hence, we conclude that a Feshbach-resonant (FR) Bose gas will always collapse in the dilute limit. Moreover, in the dilute limit, the ASF state can appear in the guise of an unstable local minimum for the free energy density, while the MSF state cannot exist. The ASF state will eventually collapse. However, within its lifetime, the properties of this system may be captured by the ASF state when we approach the unitary limit from the side with a positive scattering length. Then, we use the ϵ expansion to calculate various properties of the ASF state at $T = 0$, including the relation between the chemical potential and detuning, the equation of state, and the sound velocity. We show that the thermodynamical properties at $T = 0$ exhibit universal scaling forms near the Feshbach resonance due to the presence of a nontrivial zero-temperature, zero-density fixed point. Surprisingly, we find that the $T = 0$ thermodynamics of the ASF state in the unitary limit is exactly identical to the one for an ideal Fermi gas. This correspondence was also noticed in Refs. 8 and 9 from a different approach. Our main results are summarized in Figs. 1, 2, 3, and 4.

The rest of this paper is organized as follows: In Sec. II, we present the RG theory to fix the notation. We discuss the stability of the dilute Bose gas with a wide Feshbach resonance and its phase diagram at $T = 0$ in Sec. III. The equilibrium properties of the ASF state at $T = 0$ are calculated in Sec. IV. The last section is devoted to a conclusive discussion and comparison with previous works.

II. THE RENORMALIZATION-GROUP THEORY

We start with the action for the atom-molecule model in the grand-canonical ensemble[4]:

$$\begin{aligned} S &= \int_0^\beta d\tau \int d^d x \Psi_a^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\Psi_a \\ &+ \int_0^\beta d\tau \int d^d x \Psi_m^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} - 2\mu + \delta_0\right)\Psi_m \\ &+ \int_0^\beta d\tau \int d^d x\ U , \end{aligned} \quad (1)$$

where

$$\begin{aligned} U &= g_0\left(\Psi_m^\dagger \Psi_a \Psi_a + \mathrm{H}.c.\right) + u_3 |\Psi_a|^2 |\Psi_m|^2 \\ &+ \frac{u_1}{2}|\Psi_a|^4 + \frac{u_2}{2}|\Psi_m|^4 . \end{aligned}$$

Here Ψ_a and Ψ_m are the annihilation operators of atoms and molecules, respectively. m and M are the masses of the atom and the molecule, respectively. Galilean invariance requires that $M = 2m$. We will see later that $\delta_0 = 0$ does not correspond to the unitary limit ($a_s^{-1} \to 0$ where a_s is the scattering length between atoms) due to renormalization. Hence, we define

$$\delta = \delta_0 - \delta_c ,$$

where δ_c is a counterterm which is chosen such that $\delta = 0$ just corresponds to the unitary limit. We further define $\mu_m = 2\mu - \delta$.

We see that the action S [Eq. (1)] has a quantum critical point (QCP) at $\mu = 0 = \delta$ and $T = 1/\beta = 0$, which will be dubbed as the Gaussian fixed point. That is, S is invariant against the scaling transformation

$$\boldsymbol{x} \to \boldsymbol{x} e^{-l} , \quad \tau \to \tau e^{-2l} , \quad \Psi_{a,m} \to \Psi_{a,m} e^{dl/2} , \quad (2)$$

at this point when we take $g_0 = 0 = u_\alpha$ with $\alpha = 1, 2, 3$. We will employ this Gaussian fixed point as a point of departure to study the low-energy physics of S in the dilute limit and in the unitary regime. That is, we will work under the assumption

$$|\mu|, |\delta| \ll \frac{1}{m a_0^2} , \quad (3)$$

where a_0 is the background scattering length for atoms. We will take a_0 as the short-distance cutoff for the RG flow.

Now we would like to calculate the RG equations at $T = 0$. Under the scaling transformation (2), μ, μ_m, g_0, and u_α transform like

$$\mu' = \mu e^{2l} , \quad \mu_m' = \mu_m e^{2l} , \quad g_0' = g_0 e^{\epsilon l/2} ,$$

and

$$u_\alpha' = u_\alpha e^{(\epsilon-2)l} ,$$

where $\epsilon = 4 - d$. ($\delta_c = 0$ at the tree level.) Hence, the coupling constants u_α's are irrelevant as long as $d > 2$. However, to guarantee that the Hamiltonian of this system is bounded from below, we cannot simply set $u_\alpha = 0$. To stabilize this system, it suffices to keep the u_1 and u_2 terms. In the following, we will ignore the u_3 term[13].

To proceed, we have to calculate the one-loop RG equations for μ, μ_m, and g_0. Within the ϵ expansion, we find

$$\frac{dr_1}{dl} = 2r_1 , \quad (4)$$

$$\frac{dr_2}{dl} = 2r_2 + 2\lambda^2 r_1 - \frac{\lambda^2 r_2}{(1-r_1)^2} , \quad (5)$$

$$\frac{d\lambda}{dl} = \frac{\epsilon}{2}\lambda - \frac{\lambda^3}{2(1-r_1)^2} , \quad (6)$$

with the initial values

$$\begin{aligned} &r_1(0) = 2m\mu a_0^2 , \quad r_2(0) = 2m\mu_m a_0^2 , \\ &\lambda(0) = \sqrt{2K_d} m g_0 a_0^{\epsilon/2}, \end{aligned}$$

where $K_d = 2/[(4\pi)^{d/2}\Gamma(d/2)]$. At this order, δ_c is

$$\delta_c = 2mg_0^2 \int_{\Lambda e^{-l}}^{\Lambda} \frac{d^d p}{(2\pi)^d}\frac{1}{p^2} .$$

Scaling stops when $l = l_*$ where l_* is defined by $|r_1(l_*)| = 1$. From the solution of Eq. (4), we get

$$e^{l_*} = \frac{1}{\sqrt{2m|\mu| a_0^2}} . \quad (7)$$

The diluteness condition (the first inequality in Eq. (3)) leads to $e^{l_*} \gg 1$. Since before scaling stops $|r_1(l)| \ll 1$, Eqs. (5) and (6) can be approximated as

$$\frac{dr_2}{dl} = (2-\lambda^2) r_2 + 2\lambda^2 r_1 , \quad (8)$$

$$\frac{d\lambda}{dl} = \frac{\epsilon}{2}\lambda - \frac{\lambda^3}{2} . \quad (9)$$

We want to emphasize that his approximation does not change the fixed-point structure of Eqs. (4) – (6). In the following, we will work with Eqs. (4), (8), and (9).

For $d < 4$, Eqs. (4), (8), and (9) have two fixed points: the Gaussian fixed point $(r_\alpha, \lambda) = (0, 0)$ and the interacting zero-density fixed point $(r_\alpha, \lambda) = (0, \lambda_*)$ with $\lambda_* = \sqrt{\epsilon}$. The Gaussian fixed point is IR unstable, while the interacting zero-density fixed point is IR stable. Hence, the latter controls the low-energy physics of the dilute FR Bose gas in the unitary regime.

To relate δ to the physical detuning $\nu = -1/a_s$, we will calculate the T-matrix at zero density, and the result is

$$\begin{aligned} T^{-1}(k) &= \frac{k^2 - 2m(\delta+\delta_c)}{4mg_0^2} \\ &+ K_d m \int_0^\Lambda dq \frac{q^{d-1}}{q^2 - k^2 - i0^+} . \end{aligned}$$

Using $T(0) = 4\pi a_s/m$ and the one-loop result for δ_c, we get

$$\frac{m}{4\pi a_s} = -\frac{m\nu}{4\pi} = -\frac{\delta}{2g_0^2} \ .$$

By definition, the low-energy physics of the system with a wide Feshbach resonance at zero density is parametrized by only one parameter a_s. Hence, for wide Feshbach resonances, we may simply take $\lambda(0) = \lambda_*$ if we are only interested in the physics in the unitary regime[14], and thus we obtain

$$\delta = \frac{\nu}{bma_0} \ , \tag{10}$$

in $d = 3$ ($\epsilon = 1$) where $b = 4\pi K_3/\lambda_*^2 = 2/(\pi\lambda_*^2)$. Since $\lambda(l) = \lambda_*$ for wide Feshbach resonances, Eq. (8) can now be solved easily. Using Eq. (7) to eliminate l_*, we get

$$r_2(l_*) = 2\mathrm{sgn}(\mu) - (2ma_0^2)^{\epsilon/2}\delta|\mu|^{1-d/2} \ , \tag{11}$$

where $\mathrm{sgn}(x) = 1, -1$ for $x > 0$ and $x < 0$, respectively.

III. THE PHASE DIAGRAM AT ZERO TEMPERATURE

The free energy density f is given by

$$f = \frac{e^{-(d+2)l_*}}{2ma_0^{d+2}}\tilde{f}(l_*) \ , \tag{12}$$

where $\tilde{f}(l_*)$ is the free energy density for the renormalized Hamiltonian at $l = l_*$. Since $\lambda(l_*) = O(\epsilon)$ and $u_\alpha(l_*) \approx 0$, we may calculate $\tilde{f}(l_*)$ in terms of the perturbation theory in the coupling constants. The leading order results will be given by the mean-field theory of the renormalized Hamiltonian. Moreover, the prefactor in front of $\tilde{f}(l_*)$ in Eq. (12) is always positive, the phases of this system can be determined directly from $\tilde{f}(l_*)$.

A. Stability conditions

To proceed, we make a change of variables:

$$\boldsymbol{x} = a_0\tilde{\boldsymbol{x}} \ , \quad \tau = 2ma_0^2\tilde{\tau} \ , \quad \Psi_{a,m} = a_0^{-d/2}\tilde{\Psi}_{a,m} \ ,$$

the action S [Eq. (1)] can be written as

$$\begin{aligned} S \ = \ & \int_0^{\tilde{\beta}} d\tilde{\tau}\int d^d\tilde{x}\tilde{\Psi}_a^\dagger(\partial_{\tilde{\tau}} - \nabla^2 - r_1)\tilde{\Psi}_a \\ & + \int_0^{\tilde{\beta}} d\tilde{\tau}\int d^d\tilde{x}\tilde{\Psi}_m^\dagger\left(\partial_{\tilde{\tau}} - \frac{\nabla^2}{2} - r_2\right)\tilde{\Psi}_m \\ & + \int_0^{\tilde{\beta}} d\tilde{\tau}\int d^d\tilde{x} \ U \ , \end{aligned}$$

where $\tilde{\beta} = \beta/(2ma_0^2)$, $s_{1,2}(0) = \frac{K_d}{16}mu_{1,2}a_0^{2-d}$, and

$$U = \sqrt{\frac{2}{K_d}}\lambda\left(\tilde{\Psi}_m^\dagger\tilde{\Psi}_a^2 + \mathrm{H.}c.\right) + \frac{16s_1}{K_d}|\tilde{\Psi}_a|^4 + \frac{16s_2}{K_d}|\tilde{\Psi}_m|^4 \ .$$

At large values of $|\tilde{\Psi}_{a,m}|$, the renormalized potential U at $l = l_*$ is of the form

$$\frac{16s_1^*}{K_d}|\tilde{\Psi}_a|^4 + \frac{16s_2^*}{K_d}|\tilde{\Psi}_m|^4 \ ,$$

where $s_\alpha^* = s_\alpha(l_*)$. In order that the renormalized Hamiltonian is bounded from below, we must require that $s_{1,2}^* > 0$.

The one-loop RG equations for $s_{1,2}$ are given by

$$\frac{ds_1}{dl} \ = \ (2-d)s_1 \ , \tag{13}$$

$$\frac{ds_2}{dl} \ = \ (2-d)s_2 - \lambda^4 \ . \tag{14}$$

In Eq. (14), we have neglected the r_1 and r_2 dependence, and thus they are valid only in the dilute limit and in the unitary regime. Moreover, we do not consider the loop corrections due to the $u_{1,2}$ terms because they are irrelevant operators. For the wide Feshbach resonances, we may simply set $\lambda = \lambda_*$, and thus we get

$$\begin{aligned} s_1^* \ &= \ \frac{K_dmu_1}{16a_0^{d-2}} \ e^{(2-d)l_*} \ , \\ s_2^* \ &\approx \ \frac{K_dmu_2}{16a_0^{d-2}} \ e^{(2-d)l_*} - \frac{\epsilon^2}{d-2} \ , \end{aligned} \tag{15}$$

in the dilute limit.

The stability conditions require that $u_1 > 0$ and $u_2 > U_c$ where

$$U_c \ = \ \frac{16\epsilon^2}{(d-2)K_dm} \ (2m|\mu|)^{1-d/2} \ .$$

In other words, u_2 must be strong enough to guarantee the global stability of this system in the thermodynamical limit. For given $u_2 > 0$, the constraint $u_2 > U_c$ can also be expressed as a lower bound for $|\mu|$, i.e.

$$2m|\mu|a_0^2 > \left[\frac{16\epsilon^2a_0^{d-2}}{(d-2)K_dmu_2}\right]^{2/(d-2)} \ . \tag{16}$$

Equation (16) suggests that the FR Bose gas is stable either at moderate density or with strong dimer-dimer repulsions.

The point that the molecule-molecule repulsion is crucial to stabilize this system was also emphasized by the previous mean-field stability analysis[6]. In the $\mu, \delta < 0$ region, the mean-field theory predicts that the ASF state is stable for $u_2 > 0$[6]. Our RG analysis, however, indicates that in this region this system is thermodynamically unstable as $|\mu|$ smaller than some critical value. We notice that this conclusion is consistent with an earlier RPA calculation[15]. Within our approach, the reason behind this instability is that the molecule-molecule interaction becomes attractive at low energy even when the bare value of u_2 is positive.

B. The mean-field theory at $l = l_*$

To calculate $\tilde{f}(l_*)$, we perform a mean-field theory on the action S at $l = l_*$. By inserting the mean-field ansatz $\tilde{\Psi}_{a,m}(\tilde{\tau}, \tilde{\boldsymbol{x}}) = \Phi_{a,m}$ into S and then taking the variations with respect to $\Phi^\dagger_{a,m}$, we obtain the mean-field equations

$$- r_1^* \Phi_a + 2\sqrt{\frac{2}{K_d}} \lambda_* \Phi_a^\dagger \Phi_m + \frac{32 s_1^*}{K_d} |\Phi_a|^2 \Phi_a \; = \; 0 \; ,$$
$$-r_2^* \Phi_m + \sqrt{\frac{2}{K_d}} \lambda_* \Phi_a^2 + \frac{32 s_2^*}{K_d} |\Phi_m|^2 \Phi_m \; = \; 0 \; , \quad (17)$$

where $r^*_{1,2} = r_{1,2}(l_*)$. We will solve Eq. (17) within the ϵ expansion. From Eq. (15), we may neglect the s_1^* term and the s_2^* term is necessary only for the existence of the MSF state. Hence, there are three types of solutions for Eq. (17): the ASF state characterized by $|\Phi_a| = \sqrt{K_d r_1^* r_2^*}/(2\lambda_*)$ and $|\Phi_m| = \sqrt{K_d/8} r_1^*/\lambda_*$, the MSF state characterized by $\Phi_a = 0$ and $|\Phi_m| = \sqrt{K_d r_2^*/(32 s_2^*)}$, and the vacuum state (denoted by N) characterized by $\Phi_a = 0 = \Phi_m$. The solution for the ASF state can exist only for $r^*_{1,2} > 0$, while the solution for the MSF state can exist only for $r_2^* > 0$. In terms of these solutions, we find that $\tilde{f}_N(l_*) = 0$ and

$$\tilde{f}_{ASF}(l_*) = -\frac{K_d r_2^*}{8\epsilon} \; , \quad \tilde{f}_{MSF}(l_*) = -\frac{K_d (r_2^*)^2}{64 s_2^*} \; . \quad (18)$$

The phase diagram can be determined by comparing $\tilde{f}(l_*)$ for the ASF, MSF, and N states. A straightforward calculation gives rise to the following results: (i) The ASF state occupies the region with $r_1^* > 0$ and $0 < r_2^* < 8 s_2^*/\epsilon$; (ii) the MSF state occupies the region with $r_2^* > 8 s_2^*/\epsilon > 0$; and (iii) the rest in the r_1^*-r_2^* plane is the N state. Substituting the expressions for r_2^* and s_2^* [Eqs. (11) and (15)] into result (i), (ii), and (iii), and taking into account the stability condition [Eq. (16)], we obtain the phase diagram at $T = 0$. The ASF phase exists in the region with

$$2m\mu a_0^2 > \left[\frac{16\epsilon^2}{(d-2) K_d m u_2 a_0^{2-d}} \right]^{\frac{2}{d-2}} , \quad (19)$$

and

$$[2 - c_d(2m\mu a_0^2)](2m\mu a_0^2)^{\frac{d}{2}-1} < 2m\delta a_0^2 < 2(2m\mu a_0^2)^{\frac{d}{2}-1} \; . \quad (20)$$

The MSF phase exists in the region with

$$2m\delta a_0^2 < [2 - c_d(2m\mu a_0^2)](2m\mu a_0^2)^{d/2-1} \; , \quad (21)$$

for $2m\mu a_0^2 > \left[\frac{16\epsilon^2}{(d-2) K_d m u_2 a_0^{2-d}} \right]^{\frac{2}{d-2}}$ and

$$2m\delta a_0^2 < -2(2m|\mu| a_0^2)^{d/2-1} \; , \quad (22)$$

for $2m\mu a_0^2 < -\left[\frac{16\epsilon^2}{(d-2) K_d m u_2 a_0^{2-d}} \right]^{\frac{2}{d-2}}$. Here

$$c_d(x) = \frac{K_d m u_2 a_0^{2-d}}{2\epsilon} |x|^{d/2-1} - \frac{8\epsilon}{d-2} \; .$$

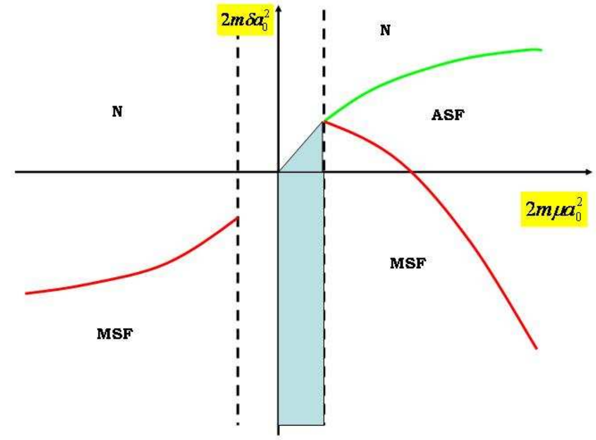

FIG. 1: (color online) The schematic phase diagram of a dilute FR Bose gas in $d = 3$. The unstable region lies between the two dashed lines. In the shaded region, the ASF state is an unstable local minimum of the free energy density.

The rest of the phase diagram is occupied by the vacuum state.

A schematic phase diagram is shown in Fig. 1. The MSF phase can exist only at finite density of atoms. Furthermore, its thermodynamical properties do not exhibit universal scaling forms even in the unitary limit ($\delta = 0$) due to the presence of a dangerously irrelevant operator u_2. This can be seen from $\tilde{f}_{MSF}(l_*)$ [Eq. (18)]. Although the thermodynamical instability in the region with $\mu > 0$ and $2m\delta a_0^2 < 2(2m\mu a_0^2)^{d/2-1}$ (the shaded region in Fig. 1) implies that the ASF state will eventually decay, the time scale, which is governed by the kinetics and dissipation, may be long enough that the ASF state appears stable. The detailed mechanism for the decay of the ASF state, however, is beyond the present study. It is hoped that the ASF state can survive for a finite lifetime if we prepare the system on the side with a positive scattering length. (After all, the experiments on ultracold atoms involve states which are thermodynamically unstable because the ground state of alkali atoms at nano-Kelvin temperatures is in fact a solid.) Finally, we must emphasize that the exact location of the unstable region (the region between the dashed lines in Fig. 1) is sensitive to the short-distance physics, and the determination of it is beyond the scope of the present approach.

IV. THE EQUILIBRIUM PROPERTIES AT ZERO TEMPERATURE

From the above analysis, the dilute FR Bose gas may be described by the ASF state on the side with a positive scattering length though the ASF state will eventually collapse. Hence, we will explore the thermodynamics of the ASF at $T = 0$.

Inserting $\tilde{f}_{ASF}(l_*)$ [Eq. (18)] into Eq. (12) and using

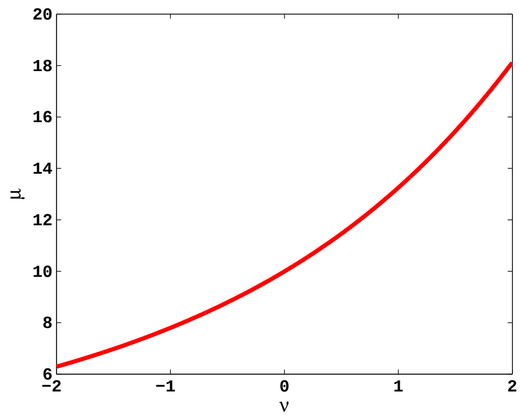

FIG. 2: (color online) The chemical potential μ (in units of $n^{2/3}/(2m)$) as a function of $\nu = -1/a_s$ (in units of $n^{1/3}$) for given atom density n.

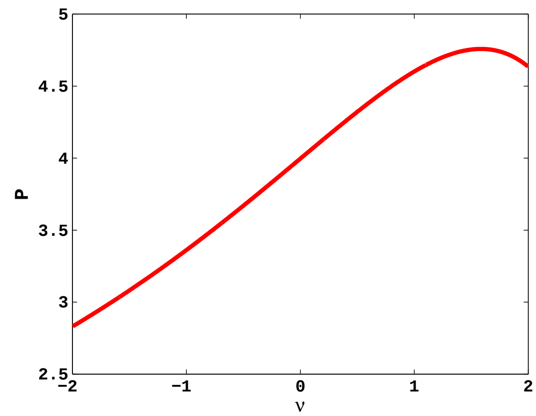

FIG. 3: (color online) The pressure P (in units of $n^{5/3}/(2m)$) as a function of $\nu = -1/a_s$ (in units of $n^{1/3}$) for given atom density n.

Eq. (11), we get the free energy density for the ASF state

$$f_{ASF} = -\frac{K_d}{4\epsilon}(2m)^{2-\frac{\epsilon}{2}}\mu^{3-\frac{\epsilon}{2}} + \frac{K_d}{8\epsilon}a_0^\epsilon\delta(2m\mu)^2\ . \tag{23}$$

From the thermodynamical relation $n = -(\partial f/\partial\mu)$, we find that

$$n = \mathcal{C}_d(2m\mu)^{\frac{d}{2}}\left\{1 - \frac{1}{3-\epsilon/2}\left[\frac{2m\delta a_0^\epsilon}{(2m\mu)^{\frac{d}{2}-1}}\right]\right\}, \tag{24}$$

where n is the density of (bare) atoms and

$$\mathcal{C}_d = \frac{K_d(3-\epsilon/2)}{4\epsilon}\ .$$

Using Euler's relation $f = -P$, we get the equation of state at $T = 0$ for the ASF phase

$$P = \frac{K_d}{4\epsilon}(2m)^{d/2}\mu^{1+d/2}\left[1 - \frac{m\delta a_0^\epsilon}{(2m\mu)^{d/2-1}}\right], \tag{25}$$

with μ as a function of n and δ given by Eq. (24). Finally, we may calculate the sound velocity v_s through the relation $v_s^2 = (n/m)\partial\mu/\partial n$, and obtain

$$v_s^2 = \frac{2\mu}{dm}\left[\frac{(3-\epsilon/2)(2m\mu)^{d/2-1} - 2m\delta a_0^\epsilon}{(3-\epsilon/2)(2m\mu)^{d/2-1} - 4m\delta a_0^\epsilon/d}\right], \tag{26}$$

with μ as a function of n and δ given by Eq. (24).

The atom density n gives rise to a characteristic length scale $n^{-1/d}$ and a characteristic energy scale $n^{2/d}/(2m)$. By measuring the lengths and energies in units of these characteristic scales, setting $d = 3$, and using Eq. (10) to replace δ by ν, Eqs. (24), (25), and (26) can be written as

$$\frac{16\pi^2}{5} = y^{3/2} - \frac{2\pi}{5}xy\ , \tag{27}$$

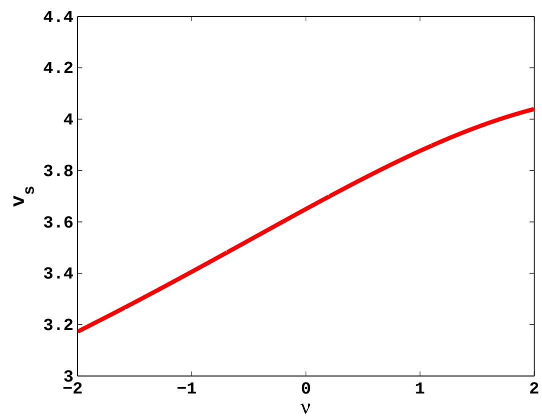

FIG. 4: (color online) The sound velocity v_s (in units of $n^{1/3}/(2m)$) as a function of $\nu = -1/a_s$ (in units of $n^{1/3}$) for given atom density n.

where $x = bm\delta a_0 n^{-1/3} = n^{-1/3}\nu$ with $\nu = -1/a_s$, $b = 2/\pi$ and $y = 2m\mu n^{-2/3}$, and

$$\tilde{P} = \frac{1}{8\pi^2}\left(y^{5/2} - \frac{\pi}{2}xy^2\right), \tag{28}$$

$$\tilde{v}_s^2 = 4y\left(\frac{5\sqrt{y} - 2\pi x}{15\sqrt{y} - 4\pi x}\right), \tag{29}$$

where $\tilde{P} = 2mn^{-5/3}P$ and $\tilde{v}_s = 2mv_s n^{-1/3}$. Equations (27) — (29) exhibit the universal scaling forms for μ, P and v_s. These universal scaling forms are due to the presence of a nontrivial zero-density fixed point. The results are shown in Figs. 2, 3, and 4.

In the unitary limit $\delta = 0$ Eqs. (24) and (25) reduce to

$$\mu = \frac{1}{2m}\left(\frac{n}{\mathcal{C}_d}\right)^{2/d}, \tag{30}$$

$$P = \frac{n^{1+2/d}}{2m(d/2+1)\mathcal{C}_d^{2/d}} \,. \qquad (31)$$

We notice that Eqs. (30) and (31) are identical to the Fermi energy and the equation of state at $T=0$ for a free Fermi gas with mass $m^* = (d\mathcal{C}_d/K_d)^{2/d}m$. That is, within the ϵ expansion, we have shown that the thermodynamics at $T=0$ for a unitary dilute Bose gas is identical to the one for a free Fermi gas. This correspondence between the thermodynamics at $T=0$ for the dilute FR Bose gas and the free Fermi gas was also noticed in Refs. 8 and 9 from an entirely different approach. Therefore, such a "fermionization" phenomenon should not be an artifact of either approach, and deserves further investigations.

V. CONCLUSIONS AND DISCUSSIONS

We study the ground-state properties of a dilute FR Bose gas from the point of view that this system can be described by a nontrivial zero-density fixed point. From a RG analysis, we first show that the FR Bose gas will always collapse in the dilute limit. Following from this consequence, the MSF state cannot exist in the dilute limit while the ASF state can survive with a finite lifetime if we prepare this system from the side with a positive scattering length. Hence, the low-energy properties of a dilute FR Base gas in the unitary regime are supposed to be described by the ASF state. Based on this observation, we show that the equilibrium properties of a dilute FR Bose gas at $T=0$ exhibit universal scaling forms near the Feshbach resonance.

Most of previous works were concentrated on the mean-field phase diagram. In the present paper, we focus on the possible universal properties of this system near the unitary limit. This point was less studied before, and our RG analysis based on the ϵ expansion may shed light on this aspect. In particular, unlike some of previous works[9,16], the present approach does not rely on the scaling hypothesis. In stead, the universality in the unitary regime naturally arises from the presence of a nontrivial zero-density fixed point. Moreover, the roles of dangerously irrelevant operators, like the u_α terms in the action, are more easily captured within the RG framework.

A striking result following from our RG analysis is that the $T=0$ thermodynamics of this system in the unitary limit is identical to that for a free Fermi gas (with a different mass). This point was also pointed out in Refs. 8 and 9 from the variational wavefunction approach. Especially, the chemical potential at zero temperature is of the form $\mu = \alpha n^{2/3}/m$ in the unitary limit, where α takes a universal value. The values of α given by Refs. 8 and 9 are 22.22 and 6.077, respectively. Since the latter is much smaller than the former, the authors of Ref. 9 claimed that the trial wavefunction adopted by them should be an energetically better candidate for the ground state. By setting $d=3$ in Eq. (30), we obtain $\alpha = 4.996$. The fact that the obtained numerical value is close to the one in Ref. 9 provides an evidence which further supports our approximate solutions based on the ϵ expansion. However, to determine the exact value of α, a more comprehensive numerical calculation is warranted. Moreover, it deserves further study whether or not such a correspondence between the dilute FR Bose gas in the unitary limit and the free Fermi gas can be extended to finite temperature and non-equilibrium processes.

The quantum phase transition between the ASF and MSF phases predicted by the mean-field theory[4] can be observed only when these phases are thermodynamically stable. According to the present analysis (Fig. 1), the ASF and MSF states can be stabilized either by strong dimer-dimer repulsions or by increasing the density of atoms to a moderate value. Therefore, to observe such a quantum phase transition, we must go beyond the dilute limit.

Our results are obtained by neglecting the higher-order correlations such as the three-body effects[17]. Therefore, they are applicable before the Efimov physics fully sets in[18,19]. It is interesting to see how the three-body effects affect the present predictions.

Acknowledgments

The work of Y.-W. Lee is supported by the National Science Council of Taiwan under grant NSC 96-2112-M-029-006-MY3. The work of Y.L. Lee is supported by the National Science Council of Taiwan under grant NSC 98-2112-M-018-003-MY3.

* Electronic address: yllee@cc.ncue.edu.tw
† Electronic address: ywlee@thu.edu.tw

1 C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); C.A. Sackett, J.M. Gerton, M. Welling, and R.G. Hulet, *ibid.*, **82**, 876 (1999); E.A. Donley, N.R. Claussen, S.L. Cornish, J.L. Roberts, E.A. Cornell, and C.E. Wieman, Nature (London) **412**, 295 (2001); J.L. Roberts, N.R. Claussen, S.L. Cornish, E.A. Donley, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **86**, 4211 (2001).

2 P.O. Fedichev, Yu. Kagan, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. **77**, 2913 (1996).

3 S. Inouye, M.R. Andrews, J. Stenger, H.-J. Miesner, D.M. Stamper-Kurn, and W. Ketterle, Nature (London) **392**, 151 (1998); S.L. Cornish, N.R. Claussen, J.L. Roberts, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **85**, 1795 (2000); N.R. Claussen, E.A. Donley, S.T. Thompson, and C.E. Wieman, *ibid.*, **89**, 010401 (2002); S.B. Papp, J.M. Pino, R.J. Wild, S. Ronen, C.E. Wieman, D.S. Jin, and E.A. Cornell, *ibid.*, **101**, 135301 (2008); S.E. Pollack, D.

Dries, M. Junker, Y.P. Chen, T.A. Corcovilos, and R.G. Hulet, *ibid.*, **102**, 090402 (2009).
[4] L. Radzihovsky, J. Park, and P.B. Weichman, Phys. Rev. Lett. **92**, 160402 (2004); M.W.J. Romans, R.A. Duine, S. Sachdev, and H.T.C. Stoof, *ibid.*. **93**, 020405 (2004); L. Radzihovsky, P.B. Weichman, and J.I. Park, Ann. Phys. **323**, 2376 (2008).
[5] Y.W. Lee and Y.L. Lee, Phys. Rev. B **70**, 224506 (2004).
[6] S. Basu and E.J. Mueller, Phys. Rev. A **78**, 053603 (2008).
[7] A. Koetsier, P. Massignan, R.A. Duine, and H.T.C. Stoof, Phys. Rev. A **79**, 063609 (2009).
[8] S. Cowell, H. Heiselberg, I.E. Mazets, J. Morales, V.R. Pandharipande, and C.J. Pethick, Phys. Rev. Lett. **88**, 210403 (2002).
[9] J.L. Song and F. Zhou, Phys. Rev. Lett. **103**, 025302 (2009).
[10] P. Nikolić and S. Sachdev, Phys. Rev. A **75**, 033608 (2007).
[11] Y. Nishida and D.T. Son, Phys. Rev. Lett. **97**, 050403 (2006); Y. Nishida and D.T. Son, Phys. Rev. A **75**, 063617 (2007); Y. Nishida, *ibid.* **75**, 063618 (2007). See also Z. Nussinov and S. Nussinov, *ibid.* **74**, 053622 (2006).
[12] M.Y. Veillette, D.E. Sheehy, and L. Radzihovsky, Phys. Rev. A **75**, 043614 (2007).
[13] The inclusion of the u_3 term will shift the boundary of the unstable region in Fig. 1, i.e. the value of the R.H.S. in Eq. (16), but does not change the qualitative structure of the phase diagram and the results we obtain.
[14] For narrow Feshbach resonances, the low-energy physics at zero density is parametrized by two parameters: the scattering length a_s and the effective range r_0. In this case, we cannot simply take $\lambda(0) = \lambda_*$. By a similar calculation for the T-matrix, we may associate δ and $\lambda(0)$ with a_s and r_0.
[15] L. Yin, Phys. Rev. A **77** 043630 (2008).
[16] T.L. Ho, Phys. Rev. Lett. **92**, 090402 (2004).
[17] B.D. Esry, C.H. Greene, and J.P. Burke, Jr., Phys. Rev. Lett. **83**, 1751 (1999); E. Braaten, H.-W. Hammer, and T. Mehen, *ibid.*, **88**, 040401 (2002).
[18] V. Efimov, Phys. Lett. **33B**, 563 (1970); Sov. J. Nucl. Phys. **12**, 589 (1971).
[19] T. Kraemer, M. Mark, P. Waldburger, J.G. Danzl, C. Chin, B. Engeser, A.D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, and R. Grimm, Nature (London) **440**, 315 (2006).